

Axiomatic approach to radiation reaction of scalar point particles in curved spacetime

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Abstract

Several different methods have recently been proposed for calculating the motion of a point particle coupled to a linearized gravitational field on a curved background. These proposals are motivated by the hope that the point particle system will accurately model certain astrophysical systems which are promising candidates for observation by the new generation of gravitational wave detectors. Because of its mathematical simplicity, the analogous system consisting of a point particle coupled to a scalar field provides a useful context in which to investigate these proposed methods. In this paper, we generalize the axiomatic approach of Quinn and Wald in order to produce a general expression for the self force on a point particle coupled to a scalar field following an arbitrary trajectory on a curved background. Our equation includes the leading order effects of the particle's own fields, commonly referred to as "self force" or "radiation reaction" effects. We then explore the equations of motion which follow from this expression in the absence of non-scalar forces.

I. INTRODUCTION

There has been much recent interest in calculating the motion of astrophysical systems which emit gravitational waves in anticipation of data from a new generation of detectors. Full three-dimensional numerical simulations are required in order to produce useful results for many of the most promising observational candidates, such as colliding black holes. However, there also exists a large class of systems which can be accurately modelled by a small isolated body moving in the fixed background created by a much larger body (e.g., a solar mass star falling into a supermassive black hole). For such a system, we might hope to produce useful results by treating the smaller object as a point particle and introducing the effects of its fields and internal structure as perturbations to the background geodesic orbit.

The perturbations due to the particle's own fields, commonly called "radiation reaction" or "self force" effects, are particularly important because they include the forces responsible

for the decay of the body’s orbit. If both the background spacetime and the unperturbed orbit of the body possess enough symmetry, it is possible to infer the effects of these forces on the orbit from global conservation principles: one calculates the energy and/or angular momentum radiated to infinity by a particle in geodesic motion, and then modifies the orbit to reflect this energy and angular momentum loss in a time-averaged fashion. (Obviously, this procedure can be iterated if greater accuracy is required.) Some justification for this method is provided by Quinn and Wald [1]. However, in the absence of such symmetries, it is necessary to directly calculate the effects of the local fields in the neighborhood of the particle. Unfortunately, this problem is ill-posed, since the fields diverge in the neighborhood of the particle’s world line, so that any such local calculation must include a rule for extracting the appropriate finite part of these divergent fields.

There is an extensive literature devoted to this regularization problem. In 1938, Dirac [2] reproduced the force expression (originally given by Abraham [3]) for a point particle coupled to an electromagnetic field in Minkowski spacetime by imposing local energy conservation on a tube surrounding the particle’s world line and subtracting the infinite contributions to the force through a “mass renormalization” scheme. In 1960, Dewitt and Brehme [4] generalized this approach to an arbitrary curved background spacetime. (A trivial calculational error in their paper was later corrected by Hobbs [5].) More recently, Mino et al. [6] further adapted this approach to produce a force expression for a point particle coupled to a linearized gravitational field on a vacuum background spacetime, and Quinn and Wald [7] rederived both the electromagnetic and gravitational forces using an axiomatic approach which, in effect, regularizes the forces by comparing forces in different spacetimes.

There has emerged from this work a consensus regarding the correct equation of motion for a particle coupled to electromagnetic fields on an arbitrary curved background and for a particle coupled to linearized gravitational fields on a vacuum background. In principle, the latter equation allows one to calculate the dynamics of the astrophysical systems of interest described above. In practice, however, very little progress has been made in applying either equation of motion to concrete physical examples for two reasons. First, given a world line in an arbitrary spacetime, the calculation of the associated retarded fields is a complex and difficult problem. Second, once these fields are calculated, the equations of motion require one to identify that portion of the retarded field at each point of the world line which arises from source contributions interior to the light cone. This part of the field is often called the “tail term,” and most approximation schemes for calculating the retarded field entangle the tail and non-tail contributions to the field.

Nevertheless, some progress has been made, notably in the electromagnetic case. In 1964, DeWitt and DeWitt [8] calculated the tail term for an electromagnetic particle in a circular orbit on a Schwarzschild background to leading order in the background curvature and the velocity of the particle. In 1980, Smith and Will [9] calculated the force on an electromagnetic particle held static on a Schwarzschild background, essentially by repeating DeWitt and Brehme’s local stress-energy conservation argument. Neither result has been generalized to the case of a massive particle coupled to gravitational fields, nor has there been any direct progress on the more complex systems which are of interest to the gravitational wave astronomy community. However, several new ideas have emerged in recent years which may lead to further progress. Ori [10] has suggested an alternative regularization scheme involving averaging of multipole moments which is better adapted to concrete calculations,

while others have suggested a hybrid scheme in which the tail term is calculated through a combination of Hadamard expansion techniques for small distances and multipole techniques for larger distances [11].

It is clearly important to test these ideas. In particular, we must know whether these schemes are equivalent to the equations of motion discussed above. Because of its mathematical simplicity, one natural system in which to explore all of these questions is that of a point particle coupled to a scalar field. Motivated by this, several researchers have begun to apply the ideas discussed above to the scalar system. In particular, Ori's method has been applied to the motion of scalar particles in the Kerr spacetime [12] and in the Schwarzschild spacetime [13,14], and Wiseman [15] has adapted the calculation of Smith and Will in order to calculate the force on a scalar particle held static in the Schwarzschild spacetime. In the present paper, we generalize the axiomatic approach of Quinn and Wald [7] in order to produce the general equation of motion for a point particle coupled to a scalar field on an arbitrary background spacetime. It is hoped that this general expression will be useful in evaluating the validity of the calculational schemes described above for the scalar case, and that this comparison will ultimately help to clarify the relationship between the various methods which have been proposed for the electromagnetic and gravitational cases.

In Sec. II, we derive an expression for the force on a particle following an arbitrary trajectory in curved spacetime. Then, in Sec. III, we explore the equations of motion which follow from this expression in the absence of non-scalar forces.

II. THE SCALAR FORCE

Given a spacetime containing a particle world line and a Klein-Gordon field sourced by the particle, we wish to define the total scalar force f_s^a on the particle at each point of the world line, including so-called self-force or radiation reaction effects. For an electromagnetic point particle in flat spacetime, an expression of this sort was first given by Abraham [3] in 1905, was later rederived in a relativistic context by Dirac [2], and is often found in textbooks (e.g., Jackson [16]). However, since there are no classical point particles in nature, and the theoretical status of such objects is problematic at best, it is important to ask how any such prescription is constrained by physics.

Our view is that the force law should reflect the force on an extended body coupled to a scalar field in the limit of small spatial extent. In particular, fix the background spacetime and consider a family of extended bodies and corresponding scalar fields parameterized by ϵ , the spatial size of the bodies. For each body in the family, we define a center of mass world line $z(\tau)$ (e.g., by the methods of Beiglböck [17]) and calculate the charge, q , and mass, m , of the body with respect to this world line,¹ as well as the force $f_s^a[\epsilon]$ exerted by the scalar field on the body. (For the definition of the force exerted on a small body by a field to

¹Because the scalar charge density is a scalar quantity, the total charge that one calculates for an extended body depends upon the spacelike surface used to slice the body. This is in contrast to electromagnetism, where the charge density is the time component of a conserved vector field and the total charge is independent of slice.

which it is coupled, see Quinn and Wald [7].) We further require that m and q vanish as ϵ goes to zero. For such a one-parameter family, Quinn and Wald [7] argue that it is possible to specify some set of conditions on the internal structure and composition of the extended bodies such that the limit of $f_S^a[\epsilon]$ for small ϵ is independent of their internal details. We would like our expression for f_S^a to correctly describe the order q and q^2 contributions to $f_S^a[\epsilon]$ which are independent of the internal structure of the body under these conditions. (Other corrections which arise from the internal structure, such as multipole effects and spin effects, have been derived elsewhere and should simply contribute additively at this order.)

Unfortunately, the limit described above is quite delicate, and the task of specifying conditions to ensure its convergence appears to be formidable. (The analysis of Dixon [18] demonstrates the degree of complexity which arises even without considering self-field effects.) Nevertheless, certain properties of this limit are strongly suggested by the nature of the divergences in the scalar field. Following Quinn and Wald [7], we will introduce these properties as axioms, and then give the unique prescription for f_S^a which satisfies these axioms.

In the next subsection, we will motivate our crucial Comparison Axiom by considering the point particle limit described above and develop the expansions required to state the axiom. Then, in the following subsection, we state both axioms and give the unique prescription for f_S^a that satisfies them, which is the main result of this paper.

A. Motivation for the Comparison Axiom

Consider a spacetime (M, g_{ab}) containing a spatially compact body characterized by stress-energy T_{body}^{ab} and scalar charge density ρ , a smooth Klein-Gordon field ϕ , and possibly some other set of fields which are coupled to the body, characterized by T_{ext}^{ab} . The Klein-Gordon field ϕ satisfies the equation

$$\nabla^a \nabla_a \phi = -4\pi \rho \quad (1)$$

with stress-energy

$$T_S^{ab} = \frac{1}{4\pi} (\nabla^a \phi \nabla^b \phi - \frac{1}{2} g^{ab} g_{cd} \nabla^c \phi \nabla^d \phi). \quad (2)$$

Assuming that the total stress-energy is conserved, so that

$$\nabla_b (T_{\text{body}}^{ab} + T_S^{ab} + T_{\text{ext}}^{ab}) = 0, \quad (3)$$

then the force density exerted on the body by the scalar field is given by

$$\nabla_b T_{\text{body}}^{ab} + \nabla_b T_{\text{ext}}^{ab} = -\nabla_b T_S^{ab} = \rho \nabla^a \phi. \quad (4)$$

Therefore, naively taking the point particle limit, we would expect the force on a scalar particle of charge q to be given by

$$f_S^a = q \nabla^a \phi. \quad (5)$$

Unfortunately, this expression is meaningless as it stands, since $\nabla_a \phi$ diverges on the world line of the particle. (The situation is exactly the same with the Lorentz force law $f_{\text{EM}}^a = qF^{ab}u_b$.) However, if we consider two points P and \tilde{P} along the world lines of two different particles in two different spacetimes (each with charge q), and we identify the neighborhoods of P and \tilde{P} , then we might hope that, under some conditions, the difference $\nabla^a \phi - \tilde{\nabla}^a \tilde{\phi}$ will be finite even as the two individual fields diverge. Under such conditions, it seems reasonable to expect that the difference between the forces on the particles will be given by the (finite) difference between the field gradients. That is,

$$f_S^a - \tilde{f}_S^a = \lim_{r \rightarrow 0} q \langle \nabla^a \phi - \tilde{\nabla}^a \tilde{\phi} \rangle_r. \quad (6)$$

(Here, the average over a sphere of radius r , denoted by $\langle \rangle_r$, is introduced to allow for the possibility that the $r \rightarrow 0$ limit of the difference is finite, but direction-dependent.)

Quinn and Wald [7] give plausibility arguments which suggest that the counterpart of Eq. (6) is indeed a property of the point particle limit in the electromagnetic and gravitational cases. These arguments generalize straightforwardly to the scalar case, so we will not give the details here. Instead, we will simply impose Eq. (6) as an axiom and investigate the consequences for f_S^a . This idea will be the basis of our crucial Comparison Axiom in the next subsection. However, first we must find out what conditions to impose on the spacetimes, the world lines near P and \tilde{P} , and the identification of their neighborhoods in order to ensure that the difference in the field gradients be finite as $r \rightarrow 0$. In order to answer this question, we will now examine in detail the singularity structure of the scalar field in the neighborhood of the world line.

Consider a scalar field satisfying Eq. (1) in a spacetime (M, g_{ab}) with a point particle source

$$\rho(x) = \int q \delta^4(x, z(\tau)) d\tau. \quad (7)$$

In contrast to the electromagnetic case, the Klein-Gordon equation does not require conservation of charge. For simplicity, we shall assume throughout our analysis that the charge q is constant along the world line. We wish to expand ϕ in r , the spatial distance from the world line $z(\tau)$. We are primarily interested in the divergent contributions to ϕ , characterized by the negative powers of r in the expansion, since these divergent contributions will determine the conditions required for convergence of the limit in Eq. (6). It follows from the general theory of propagation of singularities (see theorem 26.1.1 of Hormander [19]) that every solution of Eq. (1) which is smooth away from the world line will have the same singularity structure near the world line, so we are free to choose any convenient solution for our expansion. Later, when we wish to produce an explicit expression for f_S^a , we will want to write ϕ in terms of the advanced and retarded solutions. Therefore, these are the solutions which we will analyze in the following expansion.

Given any point x in a spacetime (M, g_{ab}) , there exists a convex normal neighborhood $C(x)$ containing x [i.e. a neighborhood $C(x)$ such that there exists a unique geodesic connecting any two points within $C(x)$]. For $x' \in C(x)$, the Hadamard elementary solution of Eq. (1) can be written in the form [4]

$$G^{(1)}(x, x') = \frac{1}{\pi} \left[\frac{U(x, x')}{\sigma(x, x')} + V(x, x') \ln |\sigma(x, x')| + W(x, x') \right], \quad (8)$$

with corresponding advanced (+) and retarded (-) Green's functions

$$G_{\pm}(x, x') = \theta_{\pm}(x, x') \left[U(x, x') \delta(\sigma(x, x')) - V(x, x') \theta(-\sigma(x, x')) \right]. \quad (9)$$

Here, $\sigma(x, x')$ is the biscalar of squared geodesic distance² and U , V , and W are all smooth biscalar fields. (For an explanation of the bitensor formalism, see Dewitt and Brehme [4].) The scalar function $\theta_{\pm}(x, x')$ is unity when x' is in the causal future/past of x and vanishes otherwise.

For x near the world line $z(\tau)$, let τ_{Σ} be the proper time along the world line which is simultaneous with x in the sense that the spatial surface Σ generated by geodesics perpendicular to u^a at $z(\tau_{\Sigma})$ intersects x . In particular, let x lie a proper distance r along the geodesic generated by unit spatial vector \hat{r}^a at $z(\tau_{\Sigma})$, and let $z(\tau_+)$ and $z(\tau_-)$ be the intersection of the world line with the future and past light cones of x , respectively. We require that x be close enough to the world line that $z(\tau_{\Sigma})$, $z(\tau_+)$, and $z(\tau_-)$ all lie within the neighborhood $C(x)$, and we denote the future and past intersections of the world line with the boundary of $C(x)$ by $z(T_+)$ and $z(T_-)$, respectively. This is illustrated in Fig. 1. For the retarded field ϕ_- , we then have

$$\begin{aligned} \phi_-(x) &= \int G_-(x, x') \rho(x') \sqrt{-g} d^4 x' \\ &= \int G_-(x, x') \left(\int q \delta^4(x', z(\tau)) d\tau \right) \sqrt{-g} d^4 x' \\ &= q \int G_-(x, z(\tau)) d\tau \\ &= q \int_{T_-}^{T_+} \theta_-[x, z(\tau)] \left[U(x, z(\tau)) \delta(\sigma(x, z(\tau))) - V(x, z(\tau)) \theta(-\sigma(x, z(\tau))) \right] d\tau \\ &\quad + q \int_{-\infty}^{T_-} G_-(x, z(\tau)) d\tau \\ &= q \int_{T_-}^{\tau_{\Sigma}} [U \delta(\sigma) - V \theta(-\sigma)] d\tau + q \int_{-\infty}^{T_-} G_- d\tau \end{aligned} \quad (10)$$

In the last line and hereafter, we suppress the spacetime dependence for all biscalars, since each depends upon x in its first argument and $z(\tau)$ in its second argument. For a bitensor A , we introduce the notation

²The biscalar of squared geodesic distance $\sigma(x, x')$ is equal to half of the squared length of the geodesic connecting x and x' : negative for timelike separated events, positive for spacelike separated events, and zero for null separated events. It is defined only when there is a unique geodesic connecting x and x' .

$$\dot{A} \equiv \frac{d}{d\tau} A(x, z(\tau)) = u^{a'} \nabla_{a'} A(x, z(\tau)). \quad (11)$$

We have

$$d\tau = \frac{d\tau}{d\sigma} d\sigma = \left(\frac{d\sigma}{d\tau} \right)^{-1} d\sigma = \dot{\sigma}^{-1} d\sigma, \quad (12)$$

which gives us

$$\phi_- = q \left[\left\{ \dot{\sigma}^{-1} U \right\}_{\tau=\tau_-} - \int_{T_-}^{\tau_-} V d\tau \right] + q \int_{-\infty}^{T_-} G_- d\tau. \quad (13)$$

We now wish to produce the corresponding expression for $\nabla_a \phi_-$. Note that the right side of Eq. (13) depends upon x in two ways: explicitly through the first argument of each biscalar and implicitly through τ_- . We have

$$\begin{aligned} \nabla_a \phi_- &= q \nabla_a \left[\left\{ \dot{\sigma}^{-1} U \right\}_{\tau=\tau_-} - \int_{T_-}^{\tau_-} V d\tau \right] + q \int_{-\infty}^{T_-} \nabla_a G_- d\tau \\ &= q \left[\left\{ -\dot{\sigma}^{-2} \nabla_a \dot{\sigma} U + \dot{\sigma}^{-1} \nabla_a U \right\}_{\tau=\tau_-} + \left\{ -\dot{\sigma}^{-2} \ddot{\sigma} U + \dot{\sigma}^{-1} \dot{U} \right\}_{\tau=\tau_-} \nabla_a \tau_- \right. \\ &\quad \left. - \int_{T_-}^{\tau_-} \nabla_a V d\tau - \{V\}_{\tau=\tau_-} \nabla_a \tau_- \right] - q \int_{-\infty}^{T_-} \nabla_a G_- d\tau \end{aligned} \quad (14)$$

Since $\sigma(x, z(\tau_-)) = 0$, we have

$$\nabla_a \{\sigma\}_{\tau=\tau_-} = \{\nabla_a \sigma\}_{\tau=\tau_-} + \{\dot{\sigma}\}_{\tau=\tau_-} \nabla_a \tau_- = 0, \quad (15)$$

so that

$$\nabla_a \tau_- = \{-\dot{\sigma}^{-1} \nabla_a \sigma\}_{\tau=\tau_-}. \quad (16)$$

Therefore, we have

$$\begin{aligned} \nabla_a \phi_- &= q \left[\left\{ -\dot{\sigma}^{-2} \nabla_a \dot{\sigma} U + \dot{\sigma}^{-1} \nabla_a U + \dot{\sigma}^{-3} \ddot{\sigma} U \nabla_a \sigma - \dot{\sigma}^{-2} \dot{U} \nabla_a \sigma + \dot{\sigma}^{-1} V \nabla_a \sigma \right\}_{\tau=\tau_-} \right. \\ &\quad \left. - \int_{T_-}^{\tau_-} \nabla_a V d\tau \right] + q \int_{-\infty}^{T_-} \nabla_a G_- d\tau \end{aligned} \quad (17)$$

In Eqs. (13) and (17), we would like to combine the integrals which appear on the right side. For $T_- \leq \tau < \tau_-$, we have $G(x, z(\tau)) = -V(x, z(\tau))$. Furthermore, since V is a smooth biscalar,

$$\int_{T_-}^{\tau_-} V d\tau = \lim_{\epsilon \rightarrow 0} \int_{T_-}^{\tau_- - \epsilon} V d\tau \quad (18)$$

and

$$\int_{T_-}^{\tau_-} \nabla_a V d\tau = \lim_{\epsilon \rightarrow 0} \int_{T_-}^{\tau_- - \epsilon} \nabla_a V d\tau. \quad (19)$$

Therefore, combining the integrals, we have

$$\phi_- = q \left\{ \dot{\sigma}^{-1} U \right\}_{\tau=\tau_-} + \lim_{\epsilon \rightarrow 0} q \int_{-\infty}^{\tau_- - \epsilon} G_- d\tau. \quad (20)$$

and

$$\begin{aligned} \nabla_a \phi_- = q \left\{ -\dot{\sigma}^{-2} \nabla_a \dot{\sigma} U + \dot{\sigma}^{-1} \nabla_a U + \dot{\sigma}^{-3} \ddot{\sigma} U \nabla_a \sigma - \dot{\sigma}^{-2} \dot{U} \nabla_a \sigma + \dot{\sigma}^{-1} V \nabla_a \sigma \right\}_{\tau=\tau_-} \\ + \lim_{\epsilon \rightarrow 0} q \int_{-\infty}^{\tau_- - \epsilon} \nabla_a G_- d\tau \end{aligned} \quad (21)$$

In order to investigate the singularity structure of ϕ_- and $\nabla_a \phi_-$ near the world line, we need expansions for the expressions in brackets on the right sides of Eqs. (20) and (21) which are valid to $O[r^0]$. (The integrals in these equations make smooth contributions to the fields.) The required small distance expansions for U , V , σ , and their derivatives can all be found in DeWitt and Brehme [4] or derived straightforwardly from expressions given therein. Switching the roles of the primed and unprimed indices for notational simplicity and including the corresponding results for the advanced field, ϕ_+ , we have

$$\phi_{\pm}(x') = q \left(r^{-1} - \frac{1}{2} a^a \hat{r}_a \right) \pm \lim_{\epsilon \rightarrow 0} q \int_{\tau_{\pm} \pm \epsilon}^{\pm \infty} G_{\pm}(x', z(\tau)) d\tau + O[r] \quad (22)$$

and

$$\begin{aligned} \nabla_{a'} \phi_{\pm}(x') = q \bar{g}_{a'a} \left(-r^{-2} \hat{r}^a - \frac{1}{2} r^{-1} a^a + \frac{1}{2} r^{-1} (a^b \hat{r}_b) \hat{r}^a - \frac{3}{8} (a^b \hat{r}_b)^2 \hat{r}^a + \frac{3}{4} (a^b \hat{r}_b) a^a \right. \\ \left. - \frac{1}{6} R_{bdce} u^b u^c \hat{r}^d \hat{r}^e \hat{r}^a - \frac{1}{8} a^2 \hat{r}^a - \frac{1}{12} R_{bc} \hat{r}^b \hat{r}^c \hat{r}^a + \frac{1}{2} (\dot{a}^b \hat{r}_b) u^a \right. \\ \left. + \frac{1}{12} R_{bc} u^b u^c \hat{r}^a + \frac{1}{6} R_{bc} u^b \hat{r}^c u^a + \frac{1}{3} R^a_{cbd} u^b u^c \hat{r}^d \pm \frac{1}{3} a^2 u^a \mp \frac{1}{3} \dot{a}^a \right. \\ \left. \mp \frac{1}{6} R_{bc} u^b u^c u^a + \frac{1}{6} R^{ab} \hat{r}_b - \frac{1}{12} R \hat{r}^a \mp \frac{1}{6} R^{ab} u_b \pm \frac{1}{12} R u^a \right) \\ \pm \lim_{\epsilon \rightarrow 0} q \int_{\tau_{\pm} \pm \epsilon}^{\pm \infty} \nabla_a G_{\pm}(x', z(\tau)) d\tau + O[r], \end{aligned} \quad (23)$$

where $\bar{g}_{a'a}$ is the bivector of geodetic parallel displacement, defined by DeWitt and Brehme [4].

We began this calculation in order to investigate what conditions we need to impose on the spacetime neighborhoods and trajectories of scalar particles in different spacetimes and on our identification of these neighborhoods in order to ensure that the subtraction of field gradients in Eq. (6) is finite, and Eq. (23) provides the answer to this question. Since the divergent terms in Eq. (23) depend only upon the four-velocity and four-acceleration of the

particle (and not, for example, on higher derivatives of the motion or the local curvature), the subtraction in Eq. (6) will be finite as long as the magnitudes of the four-accelerations of the two particles are equal and we identify the local spacetime neighborhoods in such a way that the four-velocities and four-accelerations, the geodesic distances from the world lines, and the parallel transport defined by $g_{aa'}$ all coincide up to $O[r^0]$. Given points P and \tilde{P} on two world lines such that $a^a a_a = \tilde{a}^a \tilde{a}_a$, we can achieve this by identifying the spacetime neighborhoods of P and \tilde{P} with their respective tangent spaces T_P and $T_{\tilde{P}}$ via the exponential map³, and then identifying T_P and $T_{\tilde{P}}$ via any linear map which takes u^a to \tilde{u}^a and a^a to \tilde{a}^a . Under this identification, it is clear that four-velocities, four-accelerations, and geodesic distances will coincide exactly, so we need only check that parallel transport will also agree up to the appropriate order.

One way to see this is to write out Eq. (23) explicitly in coordinates adapted to our identification map, so that each point in the neighborhood of P is mapped to the point with the same coordinates in the neighborhood of \tilde{P} . (Using such coordinates, our map identifies a vector field in the neighborhood of P with the vector field in the neighborhood of \tilde{P} having the same coordinate components.) One such coordinate system is Riemann normal coordinates⁴. In these coordinates, the coordinate components of $g_{a'a}$ are given by

$$\bar{g}_{\alpha\beta} = g_{\alpha\beta} + \frac{1}{6}r^2 R_{\alpha\gamma\beta\delta} \hat{r}^\gamma \hat{r}^\delta + O[r^3]. \quad (24)$$

(We have dropped the primed indices completely since expression relates components rather than tensors.) Comparing this to Eq. (23), we see that $\bar{g}_{\alpha\beta}$ simply acts as the identity at this order in r . [The term $-(1/6)qR_{\alpha\gamma\beta\delta}\hat{r}^\gamma\hat{r}^\delta\hat{r}^\alpha$, which arises from the multiplication of the r^{-2} term in Eq. (23) and the r^2 term in Eq. (24), vanishes by the symmetries of the Riemann tensor.] Therefore, the divergent terms will indeed cancel under the identification we have described. This provides the basis of our crucial Comparison Axiom in the next subsection.

B. The axiomatic approach

We are now prepared to give our prescription for f_S^a , the total scalar force acting on the particle. We have seen that the subtraction of field gradients in Eq. (6) will be finite as long as the two particles' four-accelerations have the same magnitude and we identify the spacetime neighborhoods via the exponential map as described above. We now elevate this property to the status of an axiom that any prescription for f_S^a must satisfy.

Axiom 1 (Comparison Axiom) Consider two points, P and \tilde{P} , each lying on time-like world lines in possibly different spacetimes which contain Klein-Gordon fields ϕ and $\tilde{\phi}$

³The exponential map identifies $v^a \in T_P$ with the spacetime point which lies unit affine parameter along the geodesic generated by v^a .

⁴In order to construct Riemann normal coordinates for a neighborhood of point P , identify points in the neighborhood with points in T_P via the exponential map, and then pick any orthonormal basis for T_P .

sourced by particles of charge q on the world lines. If the four-accelerations of the world lines at P and \tilde{P} have the same magnitude, and if we identify the neighborhoods of P and \tilde{P} via the exponential map such that the four-velocities and four-accelerations are identified, then the difference between the scalar forces f_S^a and \tilde{f}_S^a is given by the limit as $r \rightarrow 0$ of the field gradients, averaged over a sphere at geodesic distance r from the world line at P .

$$f_S^a - \tilde{f}_S^a = \lim_{r \rightarrow 0} q \langle \nabla^a \phi - \tilde{\nabla}^a \tilde{\phi} \rangle_r \quad (25)$$

Since the Comparison Axiom requires only that the four-accelerations of the particles agree, we now need only fix the dependence of f_S^a on acceleration in some arbitrary spacetime in order to uniquely determine f_S^a . Motivated by the time-reflection symmetry of the half-advanced, half-retarded solution for a uniformly accelerating trajectory in flat spacetime, we impose the following axiom, which should be familiar from electromagnetism.

Axiom 2 (Flat spacetime axiom) If (M, g_{ab}) is Minkowski spacetime, the world line is uniformly accelerating, and ϕ is the half-advanced, half-retarded solution, $\phi = \frac{1}{2}(\phi_+ + \phi_-)$, then $f_S^a = 0$ at every point on the world line.

We will now show that, if there exists a prescription for f_S^a satisfying these two axioms, it must be unique. Consider a point P on the world line of a scalar particle of charge q in some spacetime, and let the particle have acceleration a^a at point P . Let f_S^a and g_S^a be two prescriptions for the scalar force, both satisfying the axioms given above. Now consider a uniformly accelerating particle with the same charge q and the same acceleration a^a in a flat spacetime $(\mathbb{R}^4, \eta_{ab})$, and construct the half-advanced, half-retarded solution $\tilde{\phi} = \frac{1}{2}(\tilde{\phi}_+ + \tilde{\phi}_-)$ for this particle. By our second axiom, we know that $\tilde{f}_S^a = \tilde{g}_S^a = 0$ at every point \tilde{P} along the world line of this uniformly accelerating particle. Therefore, identifying the neighborhoods of P and \tilde{P} as in the Comparison Axiom above, we have

$$f_S^a - g_S^a = (f_S^a - \tilde{f}_S^a) - (g_S^a - \tilde{g}_S^a) = \lim_{r \rightarrow 0} q \langle \nabla^a \phi - \tilde{\nabla}^a \tilde{\phi} \rangle_r - \lim_{r \rightarrow 0} q \langle \nabla^a \phi - \tilde{\nabla}^a \tilde{\phi} \rangle_r = 0. \quad (26)$$

This argument establishes uniqueness, but it also demonstrates existence by providing a prescription which is guaranteed to satisfy the axioms. Namely, given a point P along the world line of a scalar particle with charge q in any spacetime, we simply construct the half-advanced, half-retarded solution $\tilde{\phi}$ for a uniformly accelerating particle in flat spacetime with the same charge and acceleration. The scalar force f_S^a is then given by

$$f_S^a = \lim_{r \rightarrow 0} q \langle \nabla^a \phi - \tilde{\nabla}^a \tilde{\phi} \rangle_r. \quad (27)$$

This is the prescription for the total scalar force which we set out to find at the beginning of this section.

Writing ϕ as $\phi = \phi_{\text{in}} + \phi_-$, we can use Eq. (23) to turn this prescription into an explicit formula for f_S^a . The result is

$$\begin{aligned} f_S^a = & q \nabla^a \phi_{\text{in}} + q^2 \left(\frac{1}{3}(\dot{a}^a - a^2 u^a) + \frac{1}{6}(R^{ab} u_b + R_{bc} u^b u^c u^a) - \frac{1}{12} R u^a \right) \\ & + \lim_{\epsilon \rightarrow 0} q^2 \int_{-\infty}^{\tau - \epsilon} \nabla^a G_-(z(\tau), z(\tau')) d\tau'. \end{aligned} \quad (28)$$

This expression, which is the main result of the paper, allows us to calculate f_S^a for any trajectory $z(\tau)$ in any spacetime. As stated at the beginning of the section, the physical significance of this expression is that it should correctly describe the order q and q^2 contributions to the force on a nearly spherical extended body in the point particle limit.

III. THE EQUATIONS OF MOTION

We now wish to consider the special case in which no non-scalar forces are present, so that the evolution of the world line $z(\tau)$ is determined by the scalar field. In the next subsection, we derive equations of motion for $z(\tau)$ in this case. Then, in the following subsection, we explore one of the consequences of these equations of motion: that the mass of particle varies with time.

A. Derivation of the equations of motion

Consider once again the extended body described in Sec. II. In the absence of non-scalar fields, conservation of stress-energy dictates that

$$\nabla_b T_{\text{body}}^{ab} = -\nabla_b T_S^{ab}. \quad (29)$$

According to the arguments of Quinn and Wald [7], in the point particle limit, the center of mass world line $z(\tau)$ will therefore satisfy

$$u^b \nabla_b (m u^a) = \frac{dm}{d\tau} u^a + m a^a = f_S^a, \quad (30)$$

where f_S^a is the limiting force we derived in Sec. II. Inserting our expression for f_S^a from Eq. (28) and separating the components parallel to u^a and perpendicular to u^a , we have

$$\begin{aligned} a^a &= \frac{1}{m} (f_S^a + u^a g_{bc} u^b f_S^c) \\ &= \frac{q}{m} (\nabla^a \phi_{\text{in}} + u^a u^b \nabla_b \phi_{\text{in}}) + \frac{q^2}{m} \left(\frac{1}{3} (\dot{a}^a - a^2 u^a) + \frac{1}{6} (R^{ab} u_b + R_{bc} u^b u^c u^a) \right) \\ &\quad + \lim_{\epsilon \rightarrow 0} \frac{q^2}{m} \int_{-\infty}^{\tau-\epsilon} (\nabla^a G_- + u^a g_{bc} u^b \nabla^c G_-) d\tau' \end{aligned} \quad (31)$$

and

$$\frac{dm}{d\tau} = -f_S^a u_a = -q u^a \nabla_a \phi_{\text{in}} - \frac{1}{12} q^2 R - \lim_{\epsilon \rightarrow 0} q^2 \int_{-\infty}^{\tau-\epsilon} u_a \nabla^a G_- d\tau'. \quad (32)$$

We now note three important features of these equations. First, for each point along the world line, the integrals in these expressions represent that portion of $\nabla^a \phi_-$ which arises from source contributions interior to the past light cone of the point. This contribution to the force, often called the “tail term,” is a direct consequence of the failure of Huygen’s principle in curved spacetime, and can be understood as the result of scalar radiation backscattering

from the background curvature and re-intersecting the particle world line. The presence of this tail term is the primary obstacle to applying these equations in physically realistic situations, since most methods for calculating the retarded field of an arbitrary world line irretrievably mix the tail and non-tail portions of the field.

Secondly, we can provide further insight into the nature of Eq. (32) by tracing the origin of the Ricci scalar term in the Hadamard expansion of the field given in Sec. II. This term arises directly from the $\{V\}_{\tau=\tau_-} \nabla_a \tau_-$ term in Eq. (14). In particular, we have

$$\lim_{\tau' \rightarrow \tau} G_-(z(\tau), z(\tau')) = \frac{1}{12} R, \quad (33)$$

so that we can rewrite Eq. (32) as

$$\frac{dm}{d\tau} = -qu^a \nabla_a (\phi_{\text{in}} + \phi_{\text{tail}}), \quad (34)$$

where ϕ_{tail} is defined by

$$\phi_{\text{tail}} = \lim_{\epsilon \rightarrow 0} q \int_{-\infty}^{\tau-\epsilon} G_- d\tau'. \quad (35)$$

The implications of Eq. (34) for global energy conservation are explored by Quinn and Wald [1].

Finally, owing to the presence of the Abraham-Lorentz \dot{a}^a term, these equations share the unphysical “runaway” solutions which have been discussed thoroughly in the electromagnetic case. (See Jackson [16] for one such discussion.) In order to interpret these solutions, it is important to remember that we view the force law given by Eq. (28) as an approximate expression for the force on an extended body, valid to $O[q^2]$, rather than a fundamental description of a point particle. Therefore, we can eliminate these unphysical solutions through the reduction of order technique. This technique is discussed in detail by Flanagan and Wald [20], but the basic idea is simple. Recall that we wish Eq. (31) to describe the limiting motion of a one-parameter family of extended bodies in which both the charge and the mass of the bodies vanish as the parameter goes to zero. For concreteness, let us assume that the charge and mass are given by $q = a\epsilon$ and $m = b\epsilon$. In order to apply the reduction of order technique to Eq. (31), we simply insert the entire right side of the equation in place of a^a in the \dot{a}^a and $a^2 u^a$ terms and discard any resulting terms which are $O[\epsilon^2]$ or higher. The result is

$$\begin{aligned} a^a = & \frac{q}{m} (\nabla^a \phi_{\text{in}} + u^a u_b \nabla^b \phi_{\text{in}}) \\ & + \frac{1}{3} \frac{q^2}{m} \left(\frac{q}{m} (u^b \nabla_b \nabla^a \phi_{\text{in}} + u^a u^b u^c \nabla_b \nabla_c \phi_{\text{in}}) - \frac{q^2}{m^2} (\nabla^b \phi_{\text{in}} \nabla_b \phi_{\text{in}} + (u^b \nabla_b \phi_{\text{in}})^2) u^a \right) \\ & + \frac{1}{6} \frac{q^2}{m} (R^{ab} u_b + R_{bc} u^b u^c u^a) + \lim_{\epsilon \rightarrow 0} \frac{q^2}{m} \int_{-\infty}^{\tau-\epsilon} (\nabla^a G_- + u^a g_{bc} \nabla^b G_- u^c) d\tau', \end{aligned} \quad (36)$$

which is free of the unphysical runaway solutions.

B. Time variation of the mass

In stark contrast to the electromagnetic case, f_S^a includes contributions which point along the four-velocity of the particle, resulting in a time-varying mass. This is not a special feature of the self force, nor of curved spacetime. Rather, it reflects a fundamental difference between the two continuum theories. Consider a small body in Minkowski spacetime with a center of mass world line $z(\tau)$. The rest mass of such a body is given by

$$m = - \int_{\Sigma} u_a T_{\text{body}}^{ab} \epsilon_{bcde}, \quad (37)$$

where u^a is the four-velocity of $z(\tau)$ (defined away from the world line by global parallelism), Σ is the surface perpendicular to u^a , and ϵ_{abcd} is the volume element compatible with the (flat) metric. Therefore, we have

$$\begin{aligned} \frac{dm}{d\tau} &= - \frac{d}{d\tau} \int_{\Sigma} u_a T_{\text{body}}^{ab} \epsilon_{bcde} \\ &= - \int_{\Sigma} \mathcal{L}_w [u_a T_{\text{body}}^{ab} \epsilon_{bcde}] \\ &= - \int_{\Sigma} u_a \nabla_b T_{\text{body}}^{ab} w^c \epsilon_{cdef}, \end{aligned} \quad (38)$$

where w^a is the vector field which connects successive time slices $\Sigma(\tau)$. For a body coupled to a scalar field, we have $\nabla_b T_{\text{body}}^{ab} = -\nabla_b T_S^{ab} = \rho \nabla^a \phi$, so that

$$\frac{dm}{d\tau} = - \int_{\Sigma} \rho u_a \nabla^a \phi w^c \epsilon_{cdef}, \quad (39)$$

which is clearly, in general, nonvanishing. By contrast, in the electromagnetic case, we have $\nabla_b T_{\text{body}}^{ab} = -\nabla_b T_{\text{EM}}^{ab} = F^{ab} j_b$, so that

$$\frac{dm}{d\tau} = - \int_{\Sigma} u_a F^{ab} j_b \phi w^c \epsilon_{cdef}. \quad (40)$$

For typical models of charged matter, j^a and u^a will become collinear as we take the point particle limit, and dm/dt will vanish.

Perhaps because it is tempting to generalize from the more familiar electromagnetic case, this time variation of the mass in the scalar case has largely been ignored in the literature. Some authors use the equation of motion $ma^a = q \nabla^a \phi$ (e.g., Shapiro and Teukolsky [21]). This equation is clearly inconsistent, and therefore in general has no solutions, since a^a is perpendicular to the four-velocity while $\nabla^a \phi$, in general, is not. Others explicitly project $\nabla^a \phi$ perpendicular to the four-velocity as in Eq. (31) above in order to obtain the acceleration of the particle, but then simply ignore the component of $\nabla^a \phi$ which points along u^a and assume that the mass is constant (e.g., Ori [10]). While such an equation of motion is mathematically consistent, it violates global conservation of stress-energy. (See Quinn and Wald [1].)

In the discussion above, we have motivated our point particle equations of motion by imposing local stress-energy conservation on continuum matter and taking the point particle limit, using our axioms to extract the appropriate finite part of the divergent fields. The time variation of the mass arises as a direct consequence of this local stress-energy conservation. In the literature on point particles, one sometimes sees an alternative derivation which makes no reference to the continuum theory. Instead, the author defines an action for the point particle system and then formally minimizes this action with respect to variations of the fields and the world line in order to produce equations of motion. For completeness, we give such a derivation here, paying particular attention to the time dependence of the particle's mass.

Fix a globally hyperbolic spacetime (M, g_{ab}) and two Cauchy surfaces for the spacetime, C_1 and C_2 . Let ϕ be a smooth scalar field and $z(\tau)$ be a smooth world line in the region V between C_1 and C_2 . We fix the value of ϕ and the position of $z(\tau)$ on C_1 and C_2 and define the action, S , as

$$S = \int_V \left[\frac{1}{8\pi} (g^{ab} \nabla_a \phi \nabla_b \phi) + \frac{1}{2} \int m g_{ab} u^a u^b \delta^4(x - z(\tau)) d\tau + \int q \phi \delta^4(x - z(\tau)) d\tau \right] \epsilon_{abcd}. \quad (41)$$

Formally minimizing this action with respect to variations of ϕ , we arrive at

$$\nabla^a \nabla_a \phi = -4\pi \int q \delta^4(x - z(\tau)) d\tau, \quad (42)$$

while minimization with respect to variations of $z(\tau)$ yields

$$\frac{dm}{d\tau} u^a + m a^a = q \nabla^a \phi. \quad (43)$$

These are the same equations we arrived at by considering the point particle limit of the continuum theory. Of course, here we have assumed ϕ and $z(\tau)$ to be smooth in order to define the action, while the solutions of Eq (42) are clearly distributional. Therefore, no solutions of these equations exist. However, we may view this as a formal derivation of our equations from an action principle.

Note that, if we had assumed from the outset that m was constant, the only change to the equations would have been to set $dm/d\tau = 0$ in Eq. (43). Clearly, the resulting equation is inconsistent, since $\nabla^a \phi$ does not, in general, point along the four-velocity. Still, one might wonder, despite the stress-energy conservation arguments given above, whether the above action can be modified to produce the equation of motion

$$\frac{dm}{d\tau} u^a + m a^a = q(\nabla^a \phi + u^a g_{bc} u^b \nabla^c \phi), \quad (44)$$

since this equation would have the immediate consequence that $dm/d\tau = 0$, as in the electromagnetic case. Wiseman [22] has considered a large class of possible coupling terms and has found that, within this class, one cannot produce Eq. (44) without introducing a nonlinear coupling on the right side of Eq. (42). Based on this work and the stress-energy considerations discussed above, we conjecture that there exists no action which produces Eq. (44) while preserving Eq. (42).

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FIGURES

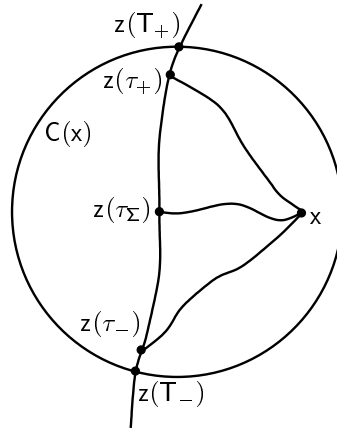


FIG. 1. The neighborhood containing x and x' .